

THE DUREN-CARLESON THEOREM IN TUBE DOMAINS OVER SYMMETRIC CONES

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ABSTRACT. In the setting of tube domains over symmetric cones, we study the characterization of the positive Borel measures μ for which the Hardy space H^p is continuously embedded into the Lebesgue space $L^q(T_\Omega, d\mu)$, $0 < p < q < \infty$. Extending a result due to Blasco for the unit disc, we reduce the problem to standard measures. We obtain that a Hardy space H^p , $1 \leq p < \infty$, embeds continuously in weighted Bergman spaces with larger exponents. Finally we use this result to characterize multipliers from H^{2m} to Bergman spaces for every positive integer m .

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Our settings are tube domains $T_\Omega = V + i\Omega$, where V is an Euclidean space of dimension n , and Ω is an irreducible symmetric cone in the complexification $V^\mathbb{C}$ of V . As in [13], r is the rank of the cone Ω while Δ is the determinant function of V . In the case $V = \mathbb{R}^n$, a typical example of an irreducible symmetric cone of rank 2 is the forward light cone Λ_n defined for $n \geq 3$ by

$$\Lambda_n = \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_1^2 - \dots - y_n^2 > 0, y_1 > 0\};$$

its determinant function is given by the Lorentz form

$$\Delta(y) = y_1^2 - \dots - y_n^2.$$

For $0 < q < \infty$ and $\nu \in \mathbb{R}$, let $L_\nu^q(T_\Omega) = L^q(T_\Omega, \Delta^{\nu-\frac{n}{r}}(y)dx dy)$ denote the space of measurable functions f satisfying the condition

$$\|f\|_{q,\nu} = \|f\|_{L_\nu^q(T_\Omega)} := \left(\int_{T_\Omega} |f(x+iy)|^q \Delta^{\nu-\frac{n}{r}}(y) dx dy \right)^{1/q} < \infty.$$

Its closed subspace consisting of holomorphic functions in T_Ω is the weighted Bergman space $A_\nu^q(T_\Omega)$. This space is not trivial i.e $A_\nu^q(T_\Omega) \neq \{0\}$ only for $\nu > \frac{n}{r} - 1$ (see [10], cf. also [1]). The Bergman projector P_ν is the orthogonal projector from the Hilbert-Lebesgue space $L_\nu^2(T_\Omega)$ to its closed subspace $A_\nu^2(T_\Omega)$. The usual (unweighted) Bergman space $A^q(T_\Omega)$ corresponds to the case $\nu = \frac{n}{r}$.

As usual, we write in the sequel $V = \mathbb{R}^n$. By $H^p(T_\Omega)$, $0 < p < \infty$, we denote the holomorphic Hardy space on the tube domain that is the space of holomorphic

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functions f such that

$$\|f\|_{H^p} = \left(\sup_{t \in \Omega} \int_{\mathbb{R}^n} |f(x + it)|^p dx \right)^{1/p} < \infty.$$

Let $0 < p < q < \infty$. Our purpose is to characterize those positive Borel measures μ on T_Ω for which the Hardy space $H^p(T_\Omega)$ is continuously imbedded into the Lebesgue space $L^q(T_\Omega, d\mu)$. We recall that given two Banach spaces of functions X and Y with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, it is said that X embeds continuously into Y ($X \hookrightarrow Y$), if there exists a constant $C > 0$ such that for any $f \in X$,

$$\|f\|_Y \leq C\|f\|_X.$$

When we test on the functions

$$G(z) = G_w(z) := [\Delta^{-\nu-\frac{n}{r}}(\frac{z-\bar{w}}{2i})]^{\frac{1}{q}},$$

with $w = u + iv \in T_\Omega$, a necessary condition is the existence of a positive constant $C_{p,q,\mu}$ such that

$$(1) \quad \int_{T_\Omega} |\Delta^{-\nu-\frac{n}{r}}(\frac{z-\bar{w}}{2i})| d\mu(z) \leq C_{p,q,\mu} \Delta^{-(\nu+\frac{n}{r})+\frac{nq}{rp}}(v)$$

whenever

$$(\nu + \frac{n}{r})\frac{p}{q} > \frac{2n}{r} - 1.$$

Our first result reduces our problem to the standard measures $d\mu(x+iy) = \Delta^{\frac{n}{r}(\frac{q}{p}-2)}(y) dx dy$ when $\frac{q}{p} > 2 - \frac{r}{n}$. We generalize to tube domains over symmetric cones a result due to O. Blasco [6] (cf. also [7]) for the unit disc.

Theorem 1.1. *Let $0 < p < q < \infty$ be such that $\frac{q}{p} > 2 - \frac{r}{n}$. Let $\nu \in \mathbb{R}$ be such that $(\nu + \frac{n}{r})\frac{p}{q} > \frac{2n}{r} - 1$. The following two assertions are equivalent.*

- (i) $H^p(T_\Omega) \hookrightarrow L^q(T_\Omega, d\mu)$ if (and only if) there exists a positive constant $C_{p,q,\mu}$ such that estimate (1) holds for every $w = u + iv \in T_\Omega$.
- (ii) $H^p(T_\Omega)$ is continuously embedded into $A_{\frac{n}{r}(\frac{q}{p}-1)}^q(T_\Omega)$.

Remark 1.2. *For $n = r = 1$ (the case of the upper half-plane, $\Omega = (0, \infty)$), assertion (i) of the theorem was proved by P. Duren [12] (cf. also [11]), using a modification of the argument given by L. Carleson [9] in the case $p = q = 2$; assertion (ii) was proved earlier by Hardy and Littlewood [14].*

In section 3, we shall prove Theorem 1.1 in a more general form where ν is a vector of \mathbb{R}^r .

Our next result is the following Hardy-Littlewood Theorem.

Theorem 1.3. *Let $4 \leq p < \infty$. Then $H^2(T_\Omega) \hookrightarrow A_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$.*

In the case where $r = 2$, it is possible to go below the power $p = 4$. We have exactly the following.

Theorem 1.4. *Let $r = 2$ and $n = 3, 4, 5, 6$. Then*

- (1) $H^2(T_{\Lambda_3}) \hookrightarrow A_{\frac{3p}{4}-\frac{3}{2}}^p(T_{\Lambda_3})$ for all $\frac{8}{3} < p < 4$.
- (2) $H^2(T_{\Lambda_4}) \hookrightarrow A_{p-2}^p(T_{\Lambda_4})$ for all $3 < p < 4$.
- (3) $H^2(T_{\Lambda_5}) \hookrightarrow A_{\frac{5p}{4}-\frac{5}{2}}^p(T_{\Lambda_5})$ for all $\frac{16}{5} < p < 4$.
- (4) $H^2(T_{\Lambda_6}) \hookrightarrow A_{\frac{3p}{2}-3}^p(T_{\Lambda_6})$ for all $\frac{10}{3} < p < 4$.

Remark 1.5. *For every positive integer $m \geq 2$, it is easy to see that the continuous embedding $H^2(T_{T_\Omega}) \hookrightarrow A_{(\frac{p}{2}-1)\frac{n}{r}}^p(T_\Omega)$ implies the continuous embedding $H^{2m}(T_{T_\Omega}) \hookrightarrow A_{(\frac{p}{2}-1)\frac{n}{r}}^{mp}(T_\Omega)$.*

Recall that given two Banach spaces of analytic functions X and Y with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, we say an analytic function G is a multiplier from X to Y , if there exists a constant $C > 0$ such that for any $F \in X$,

$$\|FG\|_Y \leq C\|F\|_X.$$

We denote by $\mathcal{M}(X, Y)$ the set of multipliers from X to Y .

Let $\alpha \in \mathbb{R}$. We denote by $H_\alpha^\infty(T_\Omega)$, the Banach space of analytic functions F on T_Ω such that

$$\|F\|_{\alpha, \infty} := \sup_{z \in T_\Omega} \Delta(\Im z)^\alpha |F(z)| < \infty.$$

In particular, for $\alpha = 0$, the space $H_0^\infty(T_\Omega)$ is the space H^∞ of bounded holomorphic functions on T_Ω . The above results allow us to obtain the following characterization of pointwise multipliers from $H^2(T_\Omega)$ to $A_\nu^p(T_\Omega)$.

Theorem 1.6. *Let $4 \leq p < \infty$, $\nu > \frac{n}{r} - 1$. Define $\gamma = \frac{1}{p}(\nu + \frac{n}{r}) - \frac{n}{2r}$. Then for any integer $m \geq 1$, the following assertions hold.*

- (a) *If $\gamma > 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A_\nu^{pm}(T_\Omega)) = H_{\frac{\gamma}{m}}^\infty(T_\Omega)$.*
- (b) *If $\gamma = 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A_\nu^{mp}(T_\Omega)) = H^\infty(T_\Omega)$.*
- (c) *If $\gamma < 0$, then $\mathcal{M}(H^2(T_\Omega), A_\nu^p(T_\Omega)) = \{0\}$*

For $p < 4$, we have under further restrictions the following.

Theorem 1.7. *Let $2(2 - \frac{r}{n}) < p < 4$, $\nu > \frac{n}{r} - 1$. Assume that $P_{(\frac{p}{2}-1)\frac{n}{r}}$ is bounded on $L_{(\frac{p}{2}-1)\frac{n}{r}}^p(T_\Omega)$. Define $\gamma = \frac{1}{p}(\nu + \frac{n}{r}) - \frac{n}{2r}$. Then for any integer $m \geq 1$, the following assertions hold.*

- (a) *If $\gamma > 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A_\nu^{mp}(T_\Omega)) = H_{\frac{\gamma}{m}}^\infty(T_\Omega)$.*
- (b) *If $\gamma = 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A_\nu^{mp}(T_\Omega)) = H^\infty(T_\Omega)$.*
- (c) *If $\gamma < 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A_\nu^{mp}(T_\Omega)) = \{0\}$*

Finally we particularize the previous problems to the tube domain over the light cone Λ_n . We take advantage of the geometry of this cone to prove the following restricted Hardy-Littlewood Theorem. The point here is that the exponent p is no more restricted to the set of even positive integers and the exponents p and q are

just related by the inequality $1 \leq p < q < \infty$. We say that a subset B of the Lorentz cone Λ_n is a restricted region with vertex at the origin O if the Euclidean distance of any point of B from O is less than a multiple of the Euclidean distance of that point from the boundary of Λ_n . We denote T_B the tube domain over B .

Theorem 1.8. *Let $1 \leq p < q < \infty$. Then given each restricted region B of the Lorentz cone Λ_n with vertex O , there exists a positive constant $C_{p,q}$ such that*

$$\int_{T_B} |F(z)|^q \Delta^{\frac{n}{2}(\frac{q}{p}-2)}(y) dx dy \leq C_{p,q,\gamma} \|F\|_{H^p}^q$$

for all $F \in H^p(T_{\Lambda_n})$.

The plan of this paper is as follows. In section 2, we present some preliminary results. The Blasco Theorem 1.1 is proved in section 3. The Hardy-Littlewood Theorems 1.3 and 1.4 are established in section 4. Section 5 is devoted to the proof of Theorem 1.6. The proof of the restricted Hardy-Littlewood Theorem 1.8 is given in section 6 while in section 7, we pose some open questions related this work.

2. PRELIMINARIES AND USEFUL RESULTS

Materials of this section are essentially from [13]. We give some definitions and useful results.

In this section, Ω is an irreducible open cone of rank r in \mathbb{R}^n . We recall that Ω induces a structure of Euclidean Jordan algebra in $V \equiv \mathbb{R}^n$, in which the closure of Ω is exactly the set $\{x^2 : x \in V\}$. Let us denote by \mathbf{e} the identity element in V and by $G(\Omega)$ the group of transformations of Ω . The canonical inner product in V is given by $(x/y) = \text{tr}(xy)$.

We denote by G the identity component in $G(\Omega)$. Since Ω is homogeneous, G acts transitively on Ω and there is a subgroup H of G which acts simply transitively on Ω . That is for every $y \in \Omega$, there exists a unique $h \in H$ such that $y = h\mathbf{e}$.

Fix a Jordan frame $\{c_1, \dots, c_r\}$ in V ; that is a systems of primitive idempotents such that

$$c_1 + \dots + c_r = \mathbf{e} \text{ and } c_i c_j = 0, \quad i \neq j.$$

Then the induced Pierce decomposition of V is

$$V = \bigoplus_{1 \leq i \leq j \leq r} V_{i,j}.$$

For $x \in V$, we denote by $\Delta_k(x)$ the determinant of the projection $P_k x$ of x , in the Jordan subalgebra $V^{(k)} = \bigoplus_{1 \leq i \leq j \leq k} V_{i,j}$. Note that $\Delta_1(x), \dots, \Delta_r(x)$ are the principal minors of $x \in V$ with respect to the above Jordan frame. It is known that for $x \in \Omega$, $\Delta_k(x) > 0$, $k = 1, \dots, r$. We observe that $\Delta = \Delta_r$. The generalized power function on Ω is defined as

$$\Delta_{\mathbf{s}}(x) = \Delta_1^{s_1-s_2}(x) \Delta_2^{s_2-s_3}(x) \dots \Delta_r^{s_r}(x), \quad \mathbf{s}(s_1, s_2, \dots, s_r) \in \mathbb{C}^r, \quad x \in \Omega.$$

We now recall the definition of the generalized gamma function on Ω :

$$\Gamma_{\Omega}(\mathbf{s}) = \int_{\Omega} e^{-\langle \mathbf{s}, \xi \rangle} \Delta_{\mathbf{s}}(\xi) \Delta^{-n/r}(\xi) d\xi \quad (\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r).$$

We set $\frac{d}{2} := \frac{\frac{n}{r}-1}{r-1}$. The above integral converges if and only if $\Re s_j > (j-1)\frac{d}{2}$, for all $j = 1, \dots, r$. Being in this case it is equal to:

$$\Gamma_{\Omega}(\mathbf{s}) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^r \Gamma(s_j - (j-1)\frac{d}{2})$$

(see Chapter VII of [13]).

In the same way we denote by $\Delta_1^*(x), \dots, \Delta_r^*(x)$ the principal minors of $x \in V$ with respect to the fixed Jordan frame $\{c_r, c_{r-1}, \dots, c_1\}$ and for $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$, we let

$$\Delta_{\mathbf{s}}^*(x) = \Delta_1^*(x)^{s_1-s_2} \Delta_2^*(x)^{s_2-s_3} \dots \Delta_r^*(x)^{s_r}.$$

We refer to [13, Proposition VII.1.2 and Proposition VII.1.6] for the following result on the Laplace transform of the generalized power function.

Lemma 2.1. *Let $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ with $\Re s_j > (j-1)\frac{d}{2}$ for all $j = 1, \dots, r$. Then, for all $y \in \Omega$ we have*

$$\int_{\Omega} e^{-\langle y, \xi \rangle} \Delta_{\mathbf{s}}(\xi) \Delta^{-\frac{n}{r}}(\xi) d\xi = \Gamma_{\Omega}(\mathbf{s}) \Delta_{-\mathbf{s}^*}^*(y)$$

with $\mathbf{s}^* = (s_r, \dots, s_1)$.

We extend the definition of the generalized power function to T_{Ω} as follows.

Definition 2.2. *For $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ such that $\Re s_j > (r-j)\frac{d}{2}$ for all $j = 1, \dots, r$. We define a holomorphic extension to T_{Ω} of the function $\Delta_{\mathbf{s}}(y)$, $y \in \Omega$, by*

$$\Delta_{-\mathbf{s}}\left(\frac{z}{i}\right) := \int_{\Omega} e^{-\langle \frac{z}{i}, \xi \rangle} \Delta_{\mathbf{s}^*}^*(\xi) \Delta^{-\frac{n}{r}}(\xi) d\xi.$$

We will be using the following definition of the β -function of the symmetric cone Ω .

$$B_{\Omega}(p, q) = \int_{\Omega \cap (\mathbf{e} - \Omega)} \Delta^{p-\frac{n}{r}}(x) \Delta^{q-\frac{n}{r}}(\mathbf{e} - x) dx,$$

where p and q are in \mathbb{C}^r . The above integral converges absolutely if $\Re p > \frac{n}{r} - 1$ and $\Re q > \frac{n}{r} - 1$, and in this case,

$$B_{\Omega}(p, q) = \frac{\Gamma_{\Omega}(p) \Gamma_{\Omega}(q)}{\Gamma_{\Omega}(p+q)}$$

(see [13, Theorem VII.1.7]).

We refer also to [13, Theorem VII.1.7] for the following result.

Lemma 2.3. *Let $p, q \in \mathbb{C}$ with $\Re p > \frac{n}{r} - 1$ and $\Re q > \frac{n}{r} - 1$. Then, for all $y \in \Omega$ we have*

$$\int_{\Omega \cap (u - \Omega)} \Delta_{p - \frac{n}{r}}(x) \Delta_{q - \frac{n}{r}}(u - x) dx = B_{\Omega}(p, q) \Delta^{p+q - \frac{n}{r}}(u).$$

The following is [1, Proposition 3.5].

Lemma 2.4. *Let $1 \leq p < \infty$ and $\nu > \frac{n}{r} - 1$. Then there is a constant $C > 0$ such that for any $f \in A_{\nu}^p(T_{\Omega})$ the following pointwise estimate holds:*

$$(2) \quad |f(z)| \leq C \Delta^{-\frac{1}{p}(\nu + \frac{n}{r})}(\Im z) \|f\|_{p, \nu}, \text{ for all } z \in T_{\Omega}.$$

We refer to [10] for the following, whose proof relies on the previous lemma.

Lemma 2.5. *Let $1 \leq p, q < \infty$, $\alpha, \beta > \frac{n}{r} - 1$. Then $A_{\alpha}^p(T_{\Omega}) \hookrightarrow A_{\beta}^q(T_{\Omega})$ if and only if $\frac{1}{p}(\alpha + \frac{n}{r}) = \frac{1}{q}(\beta + \frac{n}{r})$.*

From the above lemma, we deduce that to prove Theorem 1.3, it is enough to do this for $p = 4$.

We will make use of Paley-Wiener theory in the next section to prove Theorem 1.3 and Theorem 1.4. The following can be found in [13].

Theorem 2.6. *For every $F \in H^2(T_{\Omega})$ there exists $f \in L^2(\Omega)$ such that*

$$F(z) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Omega} e^{i(z|\xi)} f(\xi) d\xi, \quad z \in T_{\Omega}.$$

Conversely, if $f \in L^2(\Omega)$ then the integral above converges absolutely to a function $F \in H^2(T_{\Omega})$. In this case, $\|F\|_{H^2} = \|f\|_{L^2(\Omega)}$.

In the sequel, we write $V = \mathbb{R}^n$. For the proofs of the following two lemmas, cf. e.g. [15].

Lemma 2.7. *Let $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{R}^r$ and define*

$$I_{\mathbf{s}}(y) := \int_{\mathbb{R}^n} |\Delta_{-\mathbf{s}}(\frac{x + iy}{i})| dx \text{ for } y \in \mathbb{R}.$$

Then $I_{\mathbf{s}}(y)$ is finite if and only if $\Re s_j > (r - j)\frac{d}{2} + \frac{n}{r}$. In this case, $I_{\mathbf{s}}(y) = C(\mathbf{s})(\Delta_{-\mathbf{s}} \Delta^{\frac{n}{r}})(y)$.

Furthermore, the function $F(z) = F_w(z) = \Delta_{-\mathbf{s}}(\frac{z - \bar{w}}{2i})$ ($w = u + iv$ fixed in T_{Ω}) is in $H^p(T_{\Omega})$ whenever $\Re s_j > \frac{(r-j)\frac{d}{2} + \frac{n}{r}}{p}$. In this case, we have $\|F\|_{H^p} = C(s, p)(\Delta_{-\mathbf{s}} \Delta^{\frac{n}{rp}})(v)$.

Lemma 2.8. *Let $v \in T_{\Omega}$ and $\mathbf{s} = (s_1, \dots, s_r), \mathbf{t} = (t_1, \dots, t_r) \in \mathbb{C}^r$. The integral*

$$\int_{\Omega} \Delta_{-\mathbf{s}}(y + v) \Delta_{\mathbf{t}}(y) dy$$

converges if $\Re t_j > (j - 1)\frac{d}{2} - \frac{n}{r}$ et $\Re(s_j - t_j) > \frac{n}{r} + (r - j)\frac{d}{2}$. In this case this integral is equal to $C_{\mathbf{s}, \mathbf{t}}(\Delta_{-\mathbf{s} + \mathbf{t}} \Delta^{\frac{n}{r}})(v)$.

We denote as in [1]

$$L_{-\nu}^2(\Omega) = L^2(\Omega; \Delta^{-\nu}(2\xi)d\xi).$$

The following Paley-Wiener characterization of the space $A_\nu^2(T_\Omega)$ can be found in [13].

Theorem 2.9. *For every $F \in A_\nu^2(T_\Omega)$ there exists $f \in L_{-\nu}^2(\Omega)$ such that*

$$F(z) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Omega} e^{i(z|\xi)} f(\xi) d\xi, \quad z \in T_\Omega.$$

Conversely, if $f \in L_{-\nu}^2(\Omega)$ then the integral above converges absolutely to a function $F \in A_\nu^2(T_\Omega)$. In this case, $\|F\|_{p,\nu} = \|f\|_{L_{-\nu}^2}$.

The (weighted) Bergman projection P_ν is given by

$$P_\nu f(z) = \int_{T_\Omega} K_\nu(z, w) f(w) dV_\nu(w),$$

where $K_\nu(z, w) = c_\nu \Delta^{-(\nu + \frac{n}{r})}(\frac{z - \bar{w}}{2i})$ is the Bergman kernel, i.e the reproducing kernel of $A_\nu^2(T_\Omega)$ (see [13]). Here, we use the notation $dV_\nu(w) := \Delta^{\nu - \frac{n}{r}}(v) du dv$, where $w = u + iv$ is an element of T_Ω . For $\nu = \frac{n}{r}$, we simply write $dV(w)$ instead of $dV_{\frac{n}{r}}(w)$. The positive Bergman operator P_ν^+ is defined by replacing the kernel function by its modulus in the definition of P_ν .

In the particular case of the tube domain over the Lorentz cone Λ_n on \mathbb{R}^n , the following theorem is a consequence of results of [2] and the recent l^2 -decoupling theorem of [8].

Theorem 2.10. *Let $\nu > \frac{n}{2} - 1$. Then the Bergman projector P_ν of T_{Λ_n} admits a bounded extension on $L_\nu^p(T_{\Lambda_n})$ if and only if*

$$p'_\nu < p < p_\nu := \frac{\nu + n - 1}{\frac{n}{2} - 1} - \frac{(1 - \nu)_+}{\frac{n}{2} - 1}.$$

For the other cases we recall the following partial result.

Theorem 2.11. [2], [3]. *Let Ω be a symmetric cone of rank > 2 . Let $\nu > \frac{n}{r} - 1$. Then the Bergman projector P_ν of T_Ω admits a bounded extension on $L_\nu^p(\Omega)$ if*

$$q'_\nu < p < q_\nu := 2 + \frac{\nu}{\frac{n}{r} - 1}.$$

We will sometimes face situations where the weight of the projection differs from the weight associated to the space. We then need the following result (see [16]).

Proposition 2.12. *Let $1 \leq p < \infty$, $\nu \in \mathbb{R}$, and $\mu > \frac{n}{r} - 1$. Then P_μ^+ is bounded on $L_\nu^p(T_\Omega)$ if and only if $1 < p < q_\nu$ and $\mu p - \nu > (\frac{n}{r} - 1) \max\{1, p - 1\}$.*

Definition 2.13. *The generalized wave operator \square on the cone Ω is the differential operator of degree r defined by the equality*

$$\square_x[e^{i(x|\xi)}] = \Delta(\xi)e^{i(x|\xi)} \quad \text{where } \xi \in \mathbb{R}^n.$$

When applied to a holomorphic function on T_Ω , we have $\square = \square_z = \square_x$ where $z = x + iy$.

We observe with [3, 4, 1] the following.

Theorem 2.14. *Let $1 < p < \infty$ and $\nu > \frac{n}{r} - 1$.*

(1) *There exists a positive constant C such that for every $F \in A_\nu^p$,*

$$\|\square F\|_{p, \nu+p} \leq C \|F\|_{p, \nu}.$$

(2) *If moreover $p \geq 2$, the following two assertions are equivalent.*

(i) *P_ν is bounded on $L_\nu^p(T_\Omega)$;*

(ii) *For some positive integer m , the differential operator $\square^{(m)} := \square \circ \dots \circ \square$ (m times) : $A_\nu^p \rightarrow A_{\nu+mp}^p$ is a bounded isomorphism.*

Let us finish this section by the following result on complex interpolation of Bergman spaces of this setting.

Proposition 2.15. *Let $1 \leq p_0 < p_1 < \infty$, $\nu_0, \nu_1 > \frac{n}{r} - 1$. Assume that for some $\mu > \frac{n}{r} - 1$, the projection P_μ is bounded on both $L_{\nu_0}^{p_0}(T_\Omega)$ and $L_{\nu_1}^{p_1}(T_\Omega)$. Then for any $\theta \in (0, 1)$, the complex interpolation space $[A_{\nu_0}^{p_0}, A_{\nu_1}^{p_1}]_\theta$ coincides with A_ν^p with equivalent norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{\nu}{p} = \frac{1-\theta}{p_0}\nu_0 + \frac{\theta}{p_1}\nu_1$.*

Proof. Consult e.g. [5]. □

3. PROOF OF THE BLASCO THEOREM.

3.1. Proof of Theorem 1.1. By Lemma 2.7, the function

$$G(z) = G_w(z) := (\Delta_1^{-\frac{\nu_1-\nu_2}{q}} \dots \Delta_{r-1}^{-\frac{\nu_{r-1}-\nu_r}{q}} \Delta_r^{-\frac{\nu_r+\frac{n}{r}}{q}}) \left(\frac{z-\bar{w}}{2i} \right)$$

with $w = u + iv \in T_\Omega$, belongs to $H^p(T_\Omega)$ if and only if $(\nu_j + \frac{n}{r})\frac{p}{q} > (r-j)\frac{d}{2} + \frac{n}{r}$ ($j = 1, \dots, r$). Moreover

$$\|G\|_{H^p(T_\Omega)} = C_{p,q} (\Delta_1^{-\frac{\nu_1-\nu_2}{q}} \dots \Delta_{r-1}^{-\frac{\nu_{r-1}-\nu_r}{q}} \Delta_r^{-\frac{\nu_r+\frac{n}{r}}{q} + \frac{n}{rp}})(v).$$

So for these ν , a necessary condition for the continuous embedding $\mathcal{H}^p(T_\Omega) \hookrightarrow L^q(T_\Omega, d\mu)$ is the existence of a positive constant $C_{p,q,\mu}$ such that

$$(3) \quad \int_{T_\Omega} |(\Delta_1^{-(\nu_1-\nu_2)} \dots \Delta_{r-1}^{-(\nu_{r-1}-\nu_r)} \Delta_r^{-\nu_r-\frac{n}{r}}) \left(\frac{z-\bar{w}}{2i} \right)| d\mu(z) \leq C_{p,q,\mu} (\Delta_1^{-(\nu_1-\nu_2)} \dots \Delta_{r-1}^{-(\nu_{r-1}-\nu_r)} \Delta_r^{-\nu_r-\frac{n}{r}+\frac{nq}{rp}})(v)$$

for every $w = u + iv \in T_\Omega$.

We state Theorem 1.1 in the following more general form.

Theorem 3.1. *Let $0 < p < q < \infty$ be such that $\frac{q}{p} > 2 - \frac{r}{n}$. Let $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{R}^r$ be such that $(\nu_j + \frac{n}{r})\frac{p}{q} > (r-j)\frac{d}{2} + \frac{n}{r}$ ($j = 1, \dots, r$). The following two assertions are equivalent.*

- (i) $H^p(T_\Omega) \hookrightarrow L^q(T_\Omega, d\mu)$ if (and only if) there exists a positive constant $C_{p,q,\mu}$ such that the estimate (3) holds for every $w = u + iv \in T_\Omega$.
- (ii) $H^p(T_\Omega)$ is continuously embedded into $A_{\frac{n}{r}(\frac{q}{p}-1)}^q(T_\Omega)$.

Proof. We first show the implication (i) \Rightarrow (ii). We must prove that the measure $d\mu(x + iy) = \Delta^{\frac{n}{r}(\frac{q}{p}-2)}(y)dx dy$ satisfies the estimate (3), i.e.

$$\begin{aligned} L &:= \int_{T_\Omega} |(\Delta_1^{-(\nu_1-\nu_2)} \dots \Delta_{r-1}^{-(\nu_{r-1}-\nu_r)} \Delta_r^{-\nu_r-\frac{n}{r}})(\frac{z-\bar{w}}{2i})| \Delta^{\frac{n}{r}(\frac{q}{p}-2)} dx dy \\ &\leq C_{p,q,\mu} (\Delta_1^{-(\nu_1-\nu_2)} \dots \Delta_{r-1}^{-(\nu_{r-1}-\nu_r)} \Delta_r^{-\nu_r-\frac{n}{r}+\frac{nq}{rp}})(v). \end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |(\Delta_1^{-(\nu_1-\nu_2)} \dots \Delta_{r-1}^{-(\nu_{r-1}-\nu_r)} \Delta_r^{-\nu_r-\frac{n}{r}})(\frac{x+iy-\bar{w}}{2i})| dx \\ &= C_\nu \Delta_1^{-(\nu_1-\nu_2)} \dots \Delta_{r-1}^{-(\nu_{r-1}-\nu_r)} \Delta_r^{-\nu_r}(y+v) \end{aligned}$$

whenever $\nu_j > (r-j)\frac{d}{2}$ ($j = 1, \dots, r$). Moreover by Lemma 2.8, we have

$$\begin{aligned} &\int_{\Omega} (\Delta_1^{-(\nu_1-\nu_2)} \dots \Delta_{r-1}^{-(\nu_{r-1}-\nu_r)} \Delta_r^{-\nu_r}(y+v)) \Delta^{\frac{n}{r}(\frac{q}{p}-2)} dy \\ &= (\Delta_1^{-(\nu_1-\nu_2)} \dots \Delta_{r-1}^{-(\nu_{r-1}-\nu_r)} \Delta_r^{-\nu_r+\frac{n}{r}(\frac{q}{p}-1)})(v) \end{aligned}$$

whenever $\frac{q}{p} > 2 - \frac{r}{n}$ and $\nu_j > \frac{n}{r}(\frac{q}{p}-1) + (r-j)\frac{d}{2}$ ($j = 1, \dots, r$).

We next show the implication (ii) \Rightarrow (i). We shall use the following lemma.

Lemma 3.2. *Let $q > 0$ and let $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{R}^r$. There exists a positive constant $C_{q,\nu}$ such that for every $F \in \mathcal{H}ol(T_\Omega)$ we have*

$$|F(z)|^q \leq C_{q,\nu} \int_{T_\Omega} \frac{|F(u+iv)|^q (\Delta_1^{\nu_1-\nu_2} \dots \Delta_{r-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r-\frac{n}{r}})(v)}{|(\Delta_1^{\nu_1-\nu_2} \dots \Delta_{r-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r+\frac{n}{r}})(\frac{x+iy-\bar{w}}{2i})|} dudv.$$

Proof of the lemma. We denote $B(\zeta, \rho)$ the Bergman ball with centre ζ and radius ρ . Since $|F|^q$ is plurisubharmonic, we have

$$|F(ie)|^q \leq C \int_{B(ie,1)} |F(u+iv)|^q \frac{dudv}{\Delta^{\frac{2n}{r}}(v)}.$$

Recall that $\frac{dudv}{\Delta^{\frac{2n}{r}}(v)}$ is the invariant measure on T_Ω . Let $z \in T_\Omega$ and let g be an affine automorphism of T_Ω such that $g(ie) = z$. We have

$$\begin{aligned} |F(z)|^q &= |(F \circ g)(ie)|^q \\ &\leq C \int_{B(ie,1)} |(F \circ g)(u+iv)|^q \frac{dudv}{\Delta^{\frac{2n}{r}}(v)} \\ &= C \int_{B(z,1)} |F(u+iv)|^q \frac{dudv}{\Delta^{\frac{2n}{r}}(v)}. \end{aligned}$$

We recall that $|\Delta_j(\frac{z-\bar{w}}{2i})| \simeq \Delta_j(v)$ for all $w = u+iv \in B(z,1)$. This implies that

$$\begin{aligned} |F(z)|^q &\leq C_{q,\nu} \int_{B(z,1)} \frac{|F(u+iv)|^q (\Delta_1^{\nu_1-\nu_2} \dots \Delta_{r-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r-\frac{n}{r}})(v)}{|(\Delta_1^{\nu_1-\nu_2} \dots \Delta_{r-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r+\frac{n}{r}})(\frac{x+iy-\bar{w}}{2i})|} dudv \\ &\leq C_{q,\nu} \int_{T_\Omega} \frac{|F(u+iv)|^q (\Delta_1^{\nu_1-\nu_2} \dots \Delta_{r-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r-\frac{n}{r}})(v)}{|(\Delta_1^{\nu_1-\nu_2} \dots \Delta_{r-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r+\frac{n}{r}})(\frac{x+iy-\bar{w}}{2i})|} dudv. \end{aligned}$$

□

Let us set

$$I(w) := \int_{T_\Omega} \frac{d\mu(z)}{|(\Delta_1^{\nu_1-\nu_2} \dots \Delta_{r-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r+\frac{n}{r}})(\frac{x+iy-\bar{w}}{2i})|}$$

and recall that for $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{R}^r$,

$$\Delta_\nu(v) = (\Delta_1^{\nu_1-\nu_2} \dots \Delta_{r-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r-\frac{n}{r}})(v).$$

Using the Fubini-Tonelli Theorem, it follows from the previous lemma that

$$\begin{aligned} \int_{T_\Omega} |F(z)|^q d\mu(z) &\leq C_{q,\nu} \int_{T_\Omega} I(u+iv) |F(u+iv)|^q \Delta_\nu(v) dudv \\ &\leq C_{p,q,\mu} \int_{T_\Omega} \Delta^{\frac{n}{r}(\frac{q}{p}-2)}(v) |F(u+iv)|^q dudv. \end{aligned}$$

An application of the assertion (ii) of the theorem implies that

$$\int_{T_\Omega} |F(z)|^q d\mu(z) \leq C_{p,q,\nu} \|F\|_{H^p}^q.$$

This finishes the proof of the implication (ii) \Rightarrow (i).

□

4. PROOFS OF THE HARDY-LITTLEWOOD THEOREMS.

4.1. Proof of Theorem 1.4. We now give a proof of the following result which is sufficient in proving Theorem 1.3 as remarked in section 2.

Theorem 4.1. *We have that*

$$\mathcal{H}^2(T_\Omega) \hookrightarrow A^4(T_\Omega).$$

Proof. Given F in $\mathcal{H}^2(T_\Omega)$, we would like to show that F^2 belongs to $A^2(T_\Omega)$. By Theorem 2.6 there exists $f \in L^2(\Omega)$ such that

$$F(z) = \int_{\Omega} e^{i(z|\xi)} f(\xi) d\xi \quad (z \in T_\Omega).$$

It follows that

$$\begin{aligned} F^2(z) &= \int_{\Omega \times \Omega} e^{i(z|\xi+t)} f(\xi) f(t) d\xi dt \\ &= \int_{\Omega} \int_{\Omega \cap (u-\Omega)} e^{i(z|u)} f(u-\xi) f(\xi) d\xi du \\ &= \int_{\Omega} e^{i(z|u)} g(u) du, \end{aligned}$$

where

$$g(u) = \int_{\Omega \cap (u-\Omega)} f(u-\xi) f(\xi) d\xi.$$

It follows from Theorem 2.9 that to conclude, we only have to show that $g \in L^2_{-\frac{n}{r}}(\Omega)$.

We first estimate $|g(u)|^2$. Using Hölder's inequality and the definition of the beta function, we obtain

$$\begin{aligned} |g(u)|^2 &\leq \left(\int_{\Omega \cap (u-\Omega)} |f(u-\xi)| |f(\xi)| d\xi \right)^2 \\ &\leq \left(\int_{\Omega \cap (u-\Omega)} |f(u-\xi)|^2 |f(\xi)|^2 d\xi \right) \times \left(\int_{\Omega \cap (u-\Omega)} d\xi \right) \\ &= C \Delta^{\frac{n}{r}}(u) \left(\int_{\Omega \cap (u-\Omega)} |f(u-\xi)|^2 |f(\xi)|^2 d\xi \right). \end{aligned}$$

The latter equality follows from an application of Lemma 2.3. It follows easily that

$$\begin{aligned} \int_{\Omega} \Delta^{-\frac{n}{r}}(u) |g(u)|^2 du &\leq C \int_{\Omega} \int_{\Omega \cap (u-\Omega)} |f(u-\xi)|^2 |f(\xi)|^2 d\xi du \\ &= C \|f\|_{L^2(\Omega)}^4 = C \|F\|_{H^2}^4. \end{aligned}$$

The proof is complete. \square

4.2. Proof of Theorem 1.4. With an application of assertion (1) of Theorem 2.13, we deduce the following that will be useful in the proof of Theorem 1.4.

Corollary 4.2. *There exists a constant $C > 0$ such that for any $F \in H^2(T_\Omega)$,*

$$(4) \quad \left(\int_{T_\Omega} |\Delta(\Im z)(\square F)(z)|^4 dV(z) \right)^{1/4} \leq C \|F\|_{H^2(T_\Omega)}.$$

The following is also needed in our proof of Theorem 1.4.

Proposition 4.3. *For every positive integer m such that $2m > \frac{n}{r} - 1$, there exists a constant $C_m > 0$ such that for any $F \in H^2(T_\Omega)$,*

$$(5) \quad \left(\int_{T_{\Lambda_3}} |(\square^{(m)} F)(z)|^2 \Delta(\Im z)^{2m - \frac{n}{r}} dV(z) \right)^{1/2} = C \|F\|_{H^2(T_\Omega)}.$$

Proof. Let $F \in H^2(T_\Omega)$. Recall with Theorem 2.6 that there exists $f \in L^2(\Omega)$ such that

$$F(z) = \int_{\Omega} e^{i(z|\xi)} f(\xi) d\xi \quad (z \in T_\Omega)$$

with $\|F\|_{H^2(T_\Omega)} = \|f\|_{L^2(\Omega)}$. It follows that

$$\square^{(m)} F(z) = \int_{\Omega} e^{i(z|\xi)} \Delta^m(\xi) f(\xi) d\xi.$$

Using the Plancherel's formula, we obtain

$$\int_{\mathbb{R}^n} |\square^{(m)} F(x + iy)|^2 dx = \int_{\Omega} e^{-2(y|\xi)} |f(\xi)|^2 \Delta(\xi)^{2m} d\xi.$$

Integrating the latter with respect to $\Delta(y)^{2m - \frac{n}{r}} dy$ and using the definition of the gamma function, we obtain

$$\begin{aligned} I &:= \int_{T_\Omega} |\Delta^m(\Im z) (\square^{(m)} F)(z)|^2 \Delta(\Im z)^{-n/r} dV(z) \\ &= \int_{\Omega} \int_{\mathbb{R}^n} |\square^{(m)} F(x + iy)|^2 \Delta(y)^{2m - \frac{n}{r}} dy dx \\ &= \int_{\Omega} |f(\xi)|^2 \Delta(\xi)^{2m} \left(\int_{\Omega} e^{-2(y|\xi)} \Delta(y)^{2m - \frac{n}{r}} dy \right) d\xi \\ &= C_m \int_{\Omega} |f(\xi)|^2 d\xi. \end{aligned}$$

The latter equality relies on the condition $s = 2m > \frac{n}{r} - 1$ required in Lemma 2.1. \square

We use Corollary 4.2 and Proposition 4.3 to deduce the following.

Corollary 4.4. *Let $p \in (2, 4)$. For every positive integer m such that $2m > \frac{n}{r} - 1$, there exists a constant $C_m > 0$ such that for any $F \in H^2(T_\Omega)$,*

$$(6) \quad \left(\int_{T_\Omega} |(\square^{(m)} F)(z)|^p \Delta(\Im z)^{mp + (\frac{p}{2} - 2)\frac{n}{r}} dV(z) \right)^{1/p} \leq C \|F\|_{H^2(T_\Omega)}.$$

Proof. Note that by Corollary 4.2 and Proposition 4.3, $\square^{(m)}$ defines a bounded operator from $H^2(T_\Omega)$ to $A_{2m}^2(T_\Omega)$ and from $H^2(T_\Omega)$ to $A_{4m + \frac{n}{r}}^4(T_\Omega)$ respectively. It follows by interpolation that $\square^{(m)}$ is bounded from $H^2(T_\Omega)$ to $[A_{2m}^2, A_{4m + \frac{n}{r}}^4]_\theta$, $\theta \in (0, 1)$.

It is easy to check (using Proposition 2.12) that the projector $P_{4m+\frac{n}{r}}$ is bounded on both $L_{2m}^2(T_\Omega)$ and $L_{4m+\frac{n}{r}}^4(T_\Omega)$. Thus by Proposition 2.15, $[A_{2m}^2, A_{4m+\frac{n}{r}}^4]_\theta = A_{mp+(\frac{p}{2}-1)\frac{n}{r}}^p(T_\Omega)$. The proof is complete. \square

Remark 4.5. Referring to [3], we have shown that $H^2(T_\Omega)$ embeds continuously into the holomorphic Besov space $\mathbb{B}_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$ for all $2 < p < 4$.

The following follows from Corollary 4.4 and Lemma 2.14.

Theorem 4.6. Let $4 - \frac{2r}{n} < p < 4$. Assume that $P_{\frac{n}{r}(\frac{p}{2}-1)}$ is bounded on $L_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$. Then for any integer $m \geq 1$, $H^{2m}(T_\Omega) \hookrightarrow A_{\frac{n}{r}(\frac{p}{2}-1)}^{mp}(T_\Omega)$

We can now prove Theorem 1.4.

Proof of Theorem 1.4. The condition $p > 4 - \frac{2r}{n}$ is necessary for the non triviality of $A_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$. By Theorem 4.6, it is enough to check that the Bergman projector $P_{\frac{n}{r}(\frac{p}{2}-1)}$ is bounded on $L_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$. In view of Theorem 2.10 we first suppose that $r = 3$ and $\frac{n}{2}(\frac{p}{2}-1) < 1$. The inequality $\frac{n}{2}-1 < 1$ implies that $n = 3$. So we have the condition $\frac{1}{2} < \frac{3}{2}(\frac{p}{2}-1) < 1$, or equivalently $\frac{8}{3} < p < \frac{10}{3}$. By Theorem 2.10, the Bergman projector $P_{\frac{3}{2}(\frac{p}{2}-1)}$ is bounded on $L_{\frac{3}{2}(\frac{p}{2}-1)}^p(T_{\Lambda_3})$ if $4 - \frac{2r}{n} < p < 3p - 4$, or equivalently if $p > 4 - \frac{2r}{n}$. Thus the conclusion of Theorem 1.2 is valid in the case $r = 2$, $n = 3$ if $\frac{8}{3} < p < \frac{10}{3}$.

Still for $r = 2$, $n = 3$, we next suppose that $\frac{n}{2}(\frac{p}{2}-1) \geq 1$, or equivalently $p \geq \frac{10}{3}$. By Theorem 2.8, in this case, the Bergman projector $P_{\frac{3}{2}(\frac{p}{2}-1)}$ is bounded on $L_{\frac{3}{2}(\frac{p}{2}-1)}^p(T_{\Lambda_3})$ if $2 < p < \frac{3p}{2} + 1$. This condition always holds. This finishes the proof of Theorem 1.2 in the case $n = 3$.

We next suppose that $r = 2$ and $n \geq 4$. Then $\frac{n}{2}-1 \geq 1$. The condition $\frac{n}{2}(\frac{p}{2}-1) > \frac{n}{2}-1$ is equivalent to $p > \frac{4(n-1)}{n}$. Moreover by Theorem 2.10, the Bergman projector $P_{\frac{n}{2}(\frac{p}{2}-1)}$ is bounded on $L_{\frac{n}{2}(\frac{p}{2}-1)}^p(T_{\Lambda_n})$ if $\frac{4(n-1)}{n} < p < \frac{\frac{n}{2}(\frac{p}{2}-1)+n-1}{\frac{n}{2}-1}$, or equivalently if $\frac{4(n-1)}{n} < p < \frac{2n-4}{n-4}$. We must have the inequality $\frac{4(n-1)}{n} < \frac{2n-4}{n-4}$, which holds if and only if $n \leq 6$. This finishes the proof of the continuous embedding $H^2(T_\Omega) \hookrightarrow A_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$ for $n = 4, 5, 6$ respectively for all $3 < p < 4$, $\frac{16}{5} < p < 4$ and $\frac{10}{3} < p < 4$. \square

Remark 4.7. In the case $r \geq 3$, we obtain no analogous result to Theorem 1.2 using the above method. Indeed, we first observe that

$$(7) \quad n \geq \frac{r(r+1)}{2}$$

(Ω is an irreducible symmetric cone). This implies that $\frac{n}{r} \geq 2$. By Theorem 2.11, the Bergman projector $P_{\frac{n}{r}(\frac{p}{2}-1)}$ is bounded on $L_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$ if $4 - \frac{2r}{n} < p < 2 + \frac{\frac{n}{r}(\frac{p}{2}-1)}{\frac{n}{r}-1}$. This double inequality is impossible in all cases since it reduces to the inequality

$0 < 0$ if $\frac{n}{2r} = 1$ or otherwise $4 - \frac{2r}{n} < p < 2$. However the following conjecture was stated in [3].

4.3. Conjecture. For $\nu > \frac{n}{r} - 1$, the Bergman projector P_ν is bounded on $L_\nu^p(T_\Omega)$ if and only if $p'_\nu < p < p_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1}$.

Taking for granted this conjecture, the Bergman projector $P_{\frac{n}{r}(\frac{p}{2}-1)}$ is bounded on $L_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$ if $4 - \frac{2r}{n} < p < \frac{\frac{n}{r}(\frac{p}{2}-1) + \frac{2n}{r} - 1}{\frac{n}{r} - 1}$, or equivalently if $4 - \frac{2r}{n} < p < \frac{2(n-r)}{n-2r}$. The condition $4 - \frac{2r}{n} < \frac{2(n-r)}{n-2r}$ is equivalent to $3 \leq n < r(2 + \sqrt{2})$. In this case, we conclude that $H^2(T_\Omega)$ embeds continuously into $A_{\frac{n}{r}(\frac{p}{2}-1)}^p(T_\Omega)$. The inequality (7) imposes the restriction $\frac{r+1}{2} < 2 + \sqrt{2}$ or equivalently $r \in \{3, 4, 5\}$.

5. MULTIPLIERS FROM HARDY SPACES TO BERGMAN SPACES.

Let us now prove Theorem 1.6.

Proof of Theorem 1.6. We recall that $\gamma = \frac{1}{p}(\nu + \frac{n}{r}) - \frac{n}{2r}$. Let us start by proving the first assertion.

(a): First assume that $G \in H_{\frac{\gamma}{m}}^\infty(T_\Omega)$. Then using Theorem 1.3, we obtain that for any $F \in H^2(T_\Omega)$,

$$\begin{aligned} \int_{T_\Omega} |F(z)G(z)|^{mp} dV_\nu(z) &\leq \|G\|_{\frac{\gamma}{m}, \infty}^p \int_{T_\Omega} |F(z)|^{mp} \Delta(\Im z)^{(\frac{p}{2}-1)\frac{n}{r} - \frac{n}{r}} dV(z) \\ &\leq C \|G\|_{\frac{\gamma}{m}, \infty}^p \|F\|_{H^{2m}}^{mp}. \end{aligned}$$

Conversely, if $G \in \mathcal{M}(H^{2m}(T_\Omega), A_\nu^{mp}(T_\Omega))$, then by Lemma 2.4, we have a constant $C > 0$ such that for any $F \in H^{2m}(T_\Omega)$,

$$(8) \quad |F(z)G(z)| \leq C \Delta(\Im z)^{-\frac{1}{mp}(\nu + \frac{n}{r})} \|F\|_{H^{2m}}, \text{ for all } z \in T_\Omega.$$

We test (8) with the function $F(z) = F_w(z) = \Delta(\Im w)^{\frac{n}{2mr}} \Delta(\frac{z-\bar{w}}{i})^{-\frac{n}{mr}}$ (w fixed) which is uniformly in $H^{2m}(T_\Omega)$ by Lemma 2.7 and obtain that there exists $C > 0$ such that for all $z \in T_\Omega$,

$$(9) \quad |G(z)| \Delta(\Im w)^{\frac{n}{2mr}} \left| \Delta\left(\frac{z-\bar{w}}{i}\right)^{-\frac{n}{mr}} \right| \leq C \Delta(\Im z)^{-\frac{1}{mp}(\nu + \frac{n}{r})}.$$

Taking in particular $z = w$ in (9), we obtain that $\Delta(\Im w)^{\frac{1}{mp}(\nu + \frac{n}{r}) - \frac{n}{2mr}} |G(w)| \leq C$ and the constant C does not depend on w . Thus $G \in H_{\frac{\gamma}{m}}^\infty(T_\Omega)$.

(b): The proof of the necessity part follows as above. For the sufficiency, one observes that in this case, $\nu = (\frac{p}{2} - 1)\frac{n}{r}$. It follows using Theorem 1.3 that

$$\begin{aligned} \int_{T_\Omega} |F(z)G(z)|^{mp} dV_\nu(z) &\leq \|G\|_{H^\infty}^{mp} \int_{T_\Omega} |F(z)|^{mp} \Delta(\Im z)^{(\frac{p}{2}-1)\frac{n}{r} - \frac{n}{r}} dV(z) \\ &\leq C \|G\|_{H^\infty}^{mp} \|F\|_{H^{2m}}^{mp}. \end{aligned}$$

(c): It is clear that 0 is multiplier from $H^{2m}(T_\Omega)$ to $A_\nu^{mp}(T_\Omega)$. Now assume that $\gamma < 0$ and that $G \in \mathcal{M}(H^{2m}(T_\Omega), A_\nu^{mp}(T_\Omega))$. Then following exactly the same steps as in the proof of the necessity part in assertion (a), we obtain that there is a constant $C > 0$ such that for any $z \in T_\Omega$,

$$|G(z)| \leq C \Delta(\Im z)^{\frac{n}{2mr} - \frac{1}{mp}(\nu + \frac{n}{r})}.$$

As $\frac{n}{2r} - \frac{1}{p}(\nu + \frac{n}{r}) > 0$, we obtain that the right hand side of the last inequality goes to 0 as $\Delta(y) \rightarrow 0$. Hence $G(z) = 0$ for all $z \in T_\Omega$. The proof is complete. \square

Proof of Theorem 1.7. This follows as above using Theorem 4.6. \square

6. THE RESTRICTED HARDY-LITTLEWOOD THEOREM

In this section we prove Theorem 1.8. We recall that the Lorentz cone Λ_n is defined by $\Lambda_n := \{y = (y_1, y') \in \mathbb{R}^+ \times \mathbb{R}^{n-1} : y_1 > |y'|\}$. We shall rely on the following geometrical lemma.

Lemma 6.1. *We write $d\mu(y) = \frac{\Delta^\beta(y)}{y_1^{2\beta}} dy$. Then given $\beta > \frac{n}{2} - 1$, there exists a positive constant $C = C_\beta$ such that*

$$\mu(\{y \in \Lambda_n : \Delta^{\frac{n}{2}}(y) < \gamma\}) \leq C\gamma$$

for all $\gamma > 0$.

Proof of the Lemma. Using hyperbolic coordinates, an arbitrary point $y \in \Lambda_n$ can be written as

$$y = (r \operatorname{cht}, r \operatorname{sht} \omega), \quad r > 0, t \geq 0, \omega \in \mathbb{R}^{n-1}, |\omega| = 1.$$

We use spherical coordinates to write ω as

$$\omega = (\cos \varphi, \sin \varphi) \text{ with } 0 \leq \varphi \leq 2\pi \text{ if } n = 3,$$

$$\omega = (\cos \varphi_1, \sin \varphi_1 \cos \varphi_2, \dots, \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-3} \cos \varphi_{n-2}, \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-3} \sin \varphi_{n-2})$$

with $0 \leq \varphi_j \leq \pi$ ($j = 1, \dots, n-3$), $0 \leq \varphi_{n-2} \leq 2\pi$ if $n \geq 4$.

We have $r^2 = \Delta(y)$ and the Jacobian J_n of this change of coordinates has absolute value

$$\begin{aligned} |J_n| &= r^2 \operatorname{sht} t & \text{if } n = 3 \\ |J_n| &= r^{n-1} \operatorname{sht}^{n-2} t \sin^{n-3} \varphi_1 \dots \sin \varphi_{n-3} & \text{if } n \geq 4. \end{aligned}$$

Now we obtain

$$\begin{aligned} &\mu(\{y \in \Lambda_n : \Delta^{\frac{n}{2}}(y) < \gamma\}) \\ &= \int_{r^n < \gamma} r^{n-1} \frac{\operatorname{sht}^{n-2} t}{ch^{2\beta} t} \sin^{n-3} \varphi_1 \dots \sin \varphi_{n-3} dr dt d\varphi_1 \dots d\varphi_{n-3} d\varphi_{n-2} \\ &= c_n \gamma \int_0^\infty \frac{\operatorname{sht}^{n-2} t}{ch^{2\beta} t} dt. \end{aligned}$$

The latter integral converges when $\beta > \frac{n}{2} - 1$. This finishes the proof of the lemma. \square

Let $1 < p < q < \infty$ and let $\beta > \frac{n}{2} - 1$. We denote by $A_{p,\beta}^q(T_{\Lambda_n})$ the weighted Bergman space on T_{Λ_n} defined by

$$A_{p,\beta}^q(T_{\Lambda_n}) := \text{Hol}((T_{\Lambda_n}) \cap L^q(T_{\Lambda_n}, \frac{\Delta^{\frac{n}{2}(\frac{q}{p}-2)+\beta}(y)}{y_1^{2\beta}} dx dy).$$

Obviously this weighted Bergman space contains the standard weighted Bergman space $A_\nu^q(T_{\Lambda_n})$, $\nu = \frac{n}{2}(\frac{q}{p} - 1)$.

We deduce the following corollary.

Corollary 6.2. *The weighted Bergman space $A_{p,\beta}^q(T_{\Lambda_n})$ is not trivial i.e $A_{p,\beta}^q(T_{\Lambda_n}) \neq \{0\}$.*

Proof of the Corollary. We shall show that given $w = u + iv \in T_{\Lambda_n}$, the function $F(z) := \Delta^{-\frac{\nu}{q}}(\frac{z-\bar{w}}{2i})$ belongs to $A_{p,\beta}^q(T_{\Lambda_n})$ when ν is large. By Lemma 2.7 we obtain

$$\int_{\mathbb{R}^n} |F(x + iy)|^q dx = C(q, \nu) \Delta^{-\nu+\frac{n}{2}}(y + v)$$

if $\nu > n - 1$. In the notations of the previous lemma, we write again $d\mu(y) = \frac{\Delta^\beta(y)}{y_1^{2\beta}} dy$. Furthermore

$$\begin{aligned} L &:= \int_{T_{\Lambda_n}} |F(x + iy)|^q \frac{\Delta^{\frac{n}{2}(\frac{q}{p}-2)+\beta}(y)}{y_1^{2\beta}} dx dy \\ &= C(q, \nu) \int_{\Lambda_n} \Delta^{-\nu+\frac{n}{2}}(y + v) \Delta^{\frac{n}{2}(\frac{q}{p}-2)}(y) d\mu(y) \\ &= C(q, \nu) \left\{ \sum_{k=1}^{\infty} \int_{2^{-k} < \Delta^{\frac{n}{2}}(y) \leq 2^{-k+1}} + \sum_{k=0}^{\infty} \int_{2^k < \Delta^{\frac{n}{2}}(y) \leq 2^{k+1}} \right\} \end{aligned}$$

On the one hand we have

$$\begin{aligned} \sum_{k=1}^{\infty} \int_{2^{-k} < \Delta^{\frac{n}{2}}(y) \leq 2^{-k+1}} &\leq \Delta^{-\nu+\frac{n}{2}}(v) \sum_{k=1}^{\infty} \int_{2^{-k} < \Delta^{\frac{n}{2}}(y) \leq 2^{-k+1}} \Delta^{\frac{n}{2}(\frac{q}{p}-2)}(y) d\mu(y) \\ &\leq C \Delta^{-\nu+\frac{n}{2}}(v) \sum_{k=1}^{\infty} 2^{-k(\frac{q}{p}-2)} \int_{\Delta^{\frac{n}{2}}(y) \leq 2^{-k+1}} d\mu(y) \\ &\leq C_\beta \Delta^{-\nu+\frac{n}{2}}(v) \sum_{k=1}^{\infty} 2^{-k(\frac{q}{p}-1)}. \end{aligned}$$

The latter inequality follows by the previous lemma and the latter sum converges because $\frac{q}{p} > 1$. On the other hand we have

$$\begin{aligned} \sum_{k=1}^{\infty} \int_{2^k < \Delta^{\frac{n}{2}}(y) \leq 2^{k+1}} &\leq \sum_{k=1}^{\infty} \int_{2^k < \Delta^{\frac{n}{2}}(y) \leq 2^{k+1}} \Delta^{-\nu + \frac{n}{2}}(y) \Delta^{\frac{n}{2}(\frac{q}{p}-2)}(y) d\mu(y) \\ &= \sum_{k=1}^{\infty} 2^{k(-\frac{2\nu}{n} + \frac{q}{p} - 1)} \int_{\Delta^{\frac{n}{2}}(y) \leq 2^{k+1}} d\mu(y) \\ &\leq C_{\beta} \sum_{k=1}^{\infty} 2^{k(-\frac{2\nu}{n} + \frac{q}{p})}. \end{aligned}$$

The latter inequality follows by the previous lemma and the latter sum converges if ν is chosen sufficiently large. \square

We observe that for every $y \in \Lambda_n$, we have $d(y, \partial\Lambda_n) = \Delta(y)$. For the proof of Theorem 1.8 it suffices to show the following theorem.

Theorem 6.3. *Let $1 < p < q < \infty$. Then for each $\beta > \frac{n}{2} - 1$, there exists a positive constant $C_{p,q,\beta}$ such that*

$$\int_{T_{\Lambda_n}} |F(x + iy)|^q \frac{\Delta^{\frac{n}{2}(\frac{q}{p}-2)+\beta}(y)}{y_1^{2\beta}} dx dy \leq C_{p,q,\beta} \|F\|_{H^p}^q$$

for all $F \in H^p(T_{\Lambda_n})$.

Proof. In the sequel, the notation $\|\cdot\|_p$ stands for the L^p -norm in \mathbb{R}^n . We record the following well-known facts. For every $F \in H^p(T_{\Gamma_n})$, $p \geq 1$, the limit $f(x) = \lim_{y \rightarrow 0, y \in \Lambda_n} F(x + iy)$ exists in the L^p -norm; moreover if we call $P(f)$ the Poisson integral of f defined by

$$P(f)(x + iy) =: \int_{\mathbb{R}^n} \frac{\Delta^{\frac{n}{2}}(y)}{|\Delta^n(x + iy - \xi)|} f(\xi) d\xi,$$

we have $F = P(f)$ and $\|F\|_{H^p} = \|f\|_p$. So it is enough to prove that there exists a positive constant $C_{p,q,\gamma}$ such that

$$\int_{T_{\Lambda_n}} |P(f)(x + iy)|^q \frac{\Delta^{\frac{n}{2}(\frac{q}{p}-2)+\beta}(y)}{y_1^{2\beta}} dx dy \leq C_{p,q,\gamma} \|f\|_p^q.$$

We shall rely on the following lemma.

Lemma 6.4. *Given $1 \leq s \leq q < \infty$, there exists a positive constant $C_{q,s}$ such that*

$$(10) \quad \left(\int_{\mathbb{R}^n} |P(f)(x + iy)|^q dx \right)^{\frac{1}{q}} \leq C_{p,s} \|f\|_s \Delta^{\frac{n}{2}(\frac{1}{q}-\frac{1}{s})}(y)$$

for all $y \in \Lambda_n$ and $f \in L^s(\mathbb{R}^n)$.

Proof of the lemma. We apply the Young convolution inequality with the parameter $t \geq 1$ defined by $\frac{1}{q} = \frac{1}{s} + \frac{1}{t} - 1$. We obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |P(f)(x + iy)|^q dx \right)^{\frac{1}{q}} &\leq \|f\|_s \Delta^{\frac{n}{2}}(y) \left(\int_{\mathbb{R}^n} \frac{1}{|\Delta^n(x + iy)|^t} dx \right)^{\frac{1}{t}} \\ &\leq C_{q,s} \|f\|_s (\Delta^{\frac{n}{2}} \Delta^{-n+\frac{n}{2t}})(y) = C_{q,s} \|f\|_s \Delta^{\frac{n}{2}(\frac{1}{q}-\frac{1}{s})}(y). \end{aligned}$$

The latter inequality follows by Lemma 2.7. \square

We have

$$\begin{aligned} &\int_{T_{\Lambda_n}} |P(f)(x + iy)|^q \frac{\Delta^{\frac{n}{2}(\frac{q}{p}-2)+\beta}(y)}{y_1^{2\beta}} dx dy \\ &= \int_{\Lambda_n} \frac{\Delta^{\frac{n}{2}(\frac{q}{p}-2)+\beta}(y)}{y_1^{2\beta}} \left(\int_{\mathbb{R}^n} |P(f)(x + iy)|^q dx \right)^{\frac{p}{q}} \left(\int_{\mathbb{R}^n} |P(f)(x + iy)|^q dx \right)^{1-\frac{p}{q}} dy \\ &\leq C_{p,q}^p \|f\|_p^{q-p} \int_{\Lambda_n} \left(\int_{\mathbb{R}^n} |P(f)(x + iy)|^q dx \right)^{\frac{p}{q}} \frac{\Delta^{-\frac{np}{2q}+\beta}(y)}{y_1^{2\beta}} dy \end{aligned}$$

where the latter inequality follows by estimate (10) of Lemma 6.4 with $s = p$.

We define the operator S on $L^s(\mathbb{R}^n, dx)$, $1 \leq s \leq q$, by

$$Sf(y) := \Delta^{-\frac{n}{2q}}(y) \|P(f)(\cdot + iy)\|_q \quad (y \in \Lambda_n).$$

We shall show that S is a bounded operator from $L^p(\mathbb{R}^n, dx)$ to $L^p(\Lambda_n, \frac{\Delta^\beta(y)}{y_1^{2\beta}} dy)$. The conclusion will follow by the Marcinkiewicz interpolation Theorem if we can prove that S is a weak-type $(1, 1)$ operator and a weak-type (q, q) operator. The estimate (10) of Lemma 6.4 gives

$$\begin{aligned} \{y \in \Lambda_n : Sf(y) > \lambda\} &\subset \{y \in \Lambda_n : C_{p,s} \|f\|_s \Delta^{-\frac{n}{2s}} > \lambda\} \\ &= \{y \in \Lambda_n : \Delta^{\frac{n}{2}}(y) < \left(\frac{C_{p,s} \|f\|_s}{\lambda} \right)^s\}. \end{aligned}$$

An application of Lemma 6.1 concludes the proof of the theorem. \square

7. OPEN QUESTIONS

We pose here some questions that arise from this work and for which our methods do not give any answer.

- (a) Can Theorem 1.4 be extended to the interval $4 - \frac{2r}{n} < p \leq 4$ in the following two cases?
 1. $r = 2$ and $n \geq 7$;
 2. $r \geq 3$.
- (b) Can the restricted Hardy-Littlewood Theorem 1.8 be extended to the entire Lorentz cone (unrestricted)?
- (c) Can these theorems be extended to general symmetric cones?

- (d) What happens when $1 < \frac{q}{p} \leq 2 - \frac{r}{n}$ in the Duren-Carleson Theorem (equivalence (i) in Theorem 1.1)?

REFERENCES

- [1] D. BÉKOLLÉ, A. BONAMI, G. GARRIGÓS, C. NANA, M. PELOSO AND F. RICCI, *Lecture notes on Bergman projectors in tube domains over cones: an analytic and geometric viewpoint*, IMHOTEP **5** (2004), Exposé I, Proceedings of the International Workshop in Classical Analysis, Yaoundé 2001.
- [2] D. BÉKOLLÉ, A. BONAMI, G. GARRIGÓS AND F. RICCI, *Littlewood-Paley decompositions related to symmetric cones and Bergman projections in tube domains*, Proc. London Math. Soc. **89** (2004), 317-360.
- [3] D. BÉKOLLÉ, A. BONAMI, G. GARRIGÓS, F. RICCI AND B. SEHBA, *Hardy-type inequalities and analytic Besov spaces in tube domains over symmetric cones*, J. Reine Angew. Math. **647** (2010), 25-56.
- [4] D. BÉKOLLÉ, A. BONAMI, M. PELOSO AND F. RICCI, *Boundedness of weighted Bergman projections on tube domains over light cones*, Math. Z. **237** (2001), 31-59.
- [5] D. BÉKOLLÉ, J. GONESSA AND C. NANA, *Complex interpolation between two weighted Bergman spaces on tubes over symmetric cones*. C. R. Acad. Sci. Paris, Ser. I **337** (2003), 13-18.
- [6] O. BLASCO, *A remark on Carleson measures from H^p to $L^q(\mu)$ for $0 < p < q < \infty$* , Seminar of Mathematical Analysis 11-19, Colecc. Abierta, **71**, Univ. Sevilla Secr. Publ., Seville (2004)..
- [7] O. BLASCO AND H. JARCHOW, *A note on Carleson measures for Hardy spaces*, Acta Sci. Math. (Szeged) **71** (2005), No. 1-2, 371-389.
- [8] J. BOURGAIN AND C. DEMETER, *The proof of the l^2 -decoupling conjecture*, arXiv:1403.5335v3[math.CA] 26 Jul 2015.
- [9] L. CARLESON, *Interpolations by bounded analytic functions and the corona problem*, Ann. Math. (2) **76** (1962), 547-559.
- [10] D. DEBERTOL, *Besov spaces and boundedness of weighted Bergman projections over symmetric tube domains*, Dottorato di Ricerca in Matematica, Università di Genova, Politecnico di Torino, (April 2003).
- [11] P. DUREN, *Theory of H^p spaces*, Academic Press, New York (1970).
- [12] P. DUREN, *Extension of a theorem of Carleson*, Bull. Amer. Math. Soc. **75** (1969), 143-146.
- [13] J. FARAUT AND A. KORANYI, *Analysis on symmetric cones*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.
- [14] G.H. HARDY, J.E. LITTLEWOOD, *Some properties of fractional integrals II*, Math. Z. **34** (1932), 405-423.
- [15] C. NANA, *$L^{p,q}$ -Boundedness of Bergman Projections in Homogeneous Siegel Domains of Type II*, J. Fourier Anal. Appl. **19** (2013), 997-1019.
- [16] B. F. SEHBA, *Bergman type operators in tubular domains over symmetric cones*, Proc. Edin. Math. Soc. **52** (2) (2009), 529-544.

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